Solution of the radiative transfer equation in NLTE stellar atmospheres

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Outline

1. Basic equations for stellar atmospheres
2. Radiative transfer equation
3. Equations of statistical equilibrium
4. Discretization
5. Formal solution of RTE
6. Lambda iteration
7. Complete linearization
8. Accelerated lambda iteration

Role of radiation in stellar atmospheres

- source of information about star and stellar atmosphere
- influence on matter in stellar atmosphere
  - non-local (long distance) interaction (photon mean free path $\gg$ particle mean free path)
  - change of the population numbers (non-equilibrium values)
  - radiatively driven stellar wind

Basic equations

- assume
  - static atmosphere ($\vec{v} = 0$)
  - stationary atmosphere ($\partial/\partial t = 0$)
  - 1-dimensional atmosphere
  - given temperature structure $T(\vec{r})$
  - given density structure $\rho(\vec{r})$
- equations to be solved (numerically):
  - radiative transfer equation
  - equations of statistical equilibrium

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  - radiative transfer equation
  - equations of statistical equilibrium
- account for
  - full angle and frequency dependence ($I\,(\mu,\nu) \equiv I_{\mu\nu}$)

Radiative transfer equation

$$\mu \frac{\mathrm{d}I_{\mu\nu}(z)}{\mathrm{d}z} = \eta_\nu(z) - \chi_\nu(z) I_{\mu\nu}(z) = \chi_\nu \left[S_\nu(z) - I_{\mu\nu}(z)\right]$$

$\eta_\nu(z)$ – emissivity

$\chi_\nu(z)$ – opacity

$S_\nu(z) = \frac{\eta_\nu(z)}{\chi_\nu(z)}$ – source function

Radiative transfer equation

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introduce

$\mathrm{d}\tau_\nu = -\chi_\nu \, \mathrm{d}z$ – optical depth

$J_\nu = \frac{1}{2} \int_{-1}^{1} I_{\mu\nu} \, \mathrm{d}\mu$ – mean intensity

$K_\nu = \frac{1}{2} \int_{-1}^{1} \mu^2 I_{\mu\nu} \, \mathrm{d}\mu$

$f_\nu = K_\nu / J_\nu$ – variable Eddington factor (Auer & Mihalas 1970)

Radiative transfer equation

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2nd order equation

$$\frac{\mathrm{d}^2 \left[f_\nu(z) J_\nu(z) \right]}{\mathrm{d}\tau_\nu^2} = J_\nu(z) - S_\nu(z)$$

Radiative transfer equation

2nd order equation

$$\frac{\mathrm{d}^2\left[f_\nu(z)J_\nu(z)\right]}{\mathrm{d}\tau_\nu^2} = J_\nu(z) - S_\nu(z)$$

$+$ boundary conditions

$$\frac{\mathrm{d}\left[f_\nu(z)J_\nu(z)\right]}{\mathrm{d}\tau_\nu} = g_\nu(z) - H_\nu^- \qquad \text{upper}$$

$$\frac{\mathrm{d}\left[f_\nu(z)J_\nu(z)\right]}{\mathrm{d}\tau_\nu} = H_\nu^+ + g_\nu(z) \qquad \text{lower}$$

$$g_\nu = \frac{\int_0^1 \mu\left(I_{\mu\nu}^+ - I_{\mu\nu}^-\right)\mathrm{d}\mu}{J_\nu}$$

Radiative transfer equation

opacity

$$\chi_\nu = \sum_i \sum_{l>i} \left[n_i - \frac{g_i}{g_l} n_l \right] \alpha_{il}(\nu) + \sum_i \left(n_i - n_i^* e^{-\frac{h\nu}{kT}} \right) \alpha_{ik}(\nu) +$$

$$\sum_k n_e n_k \alpha_{kk}(\nu, T) \left(1 - e^{-\frac{h\nu}{kT}} \right) + n_e \sigma_e$$

emissivity

$$\eta_\nu = \frac{2h\nu^3}{c^2} \left[\sum_i \sum_{l>i} n_l \frac{g_i}{g_l} \alpha_{il}(\nu) + \sum_i n_i^* \alpha_{ik}(\nu) e^{-\frac{h\nu}{kT}} + \right.$$

$$\left. \sum_k n_e n_k \alpha_{kk}(\nu, T) e^{-\frac{h\nu}{kT}} \right]$$

Radiative transfer equation

opacity

$$\chi_\nu = \sum_i \sum_{l>i} \left[n_i^* - \frac{g_i}{g_l} n_l^* \right] \alpha_{il}(\nu) + \sum_i \left(n_i^* - n_i^* e^{-\frac{h\nu}{kT}} \right) \alpha_{ik}(\nu) +$$
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$$\left. \sum_k n_e n_k \alpha_{kk}(\nu, T) e^{-\frac{h\nu}{kT}} \right]$$

Equations of statistical equilibrium

in LTE: $n_i^* = f\,(n_e, T)$ – Saha-Boltzmann distribution

Equations of statistical equilibrium

outside LTE: $n_i = f\left(n_e, T, J_\nu\right)$ – statistical equilibrium

for $i = 1, \ldots \mathrm{NL}$

$$\sum_{l \neq i} \left\{ n_l \left[R_{li} + C_{li} \right] - n_i \left(R_{il} + C_{il} \right) \right\} = 0$$

Equations of statistical equilibrium

outside LTE: $n_i = f\left(n_e, T, J_\nu\right)$ – statistical equilibrium

for $i = 1, \ldots \mathrm{NL}$

$$\sum_{l \neq i} \left\{n_l \left[R_{li} + C_{li}\right] - n_i \left(R_{il} + C_{il}\right)\right\} = 0$$

collisional rates

$$n_i C_{il} = n_i n_e q_{il}(T),$$

Equations of statistical equilibrium

outside LTE: $n_i = f\left(n_e, T, J_\nu\right)$ – statistical equilibrium

for $i = 1, \ldots \mathrm{NL}$

$$\sum_{l \neq i} \left\{ n_l \left[R_{li} + C_{li}\right] - n_i \left(R_{il} + C_{il}\right)\right\} = 0$$

radiative rates

$$n_i R_{il} = n_i 4\pi \int \frac{\alpha_{il}(\nu)}{h\nu} J_\nu \, \mathrm{d}\nu$$

$$n_l R_{li} = n_l \frac{g_i}{g_j} 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu\right) \mathrm{d}\nu$$

Equations of statistical equilibrium

outside LTE: $n_i = f\left(n_e, T, J_\nu\right)$ – statistical equilibrium

for $i = 1, \ldots \mathrm{NL}$

$$\sum_{l \neq i} \left\{ n_l \left[R_{li}(J_\nu) + C_{li} \right] - n_i \left(R_{il}(J_\nu) + C_{il} \right) \right\} = 0$$

radiative rates

$$n_i R_{il} = n_i 4\pi \int \frac{\alpha_{il}(\nu)}{h\nu} J_\nu \, \mathrm{d}\nu$$

$$n_l R_{li} = n_l \frac{g_i}{g_j} 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu \right) \mathrm{d}\nu$$

Final radiative transfer equation

$$\frac{\mathrm{d}^2\left[f_\nu(z)J_\nu(z)\right]}{\mathrm{d}\tau_\nu^2} = J_\nu(z) - S_\nu(z, J_{\nu'})$$

$$S_\nu = \frac{\eta_\nu}{\chi_\nu}$$

$$f_\nu = \frac{K_\nu}{J_\nu} \qquad \left(K_\nu = \int_{-1}^{1} \mu^2 I_{\mu\nu}\,\mathrm{d}\mu\right)$$

$$\chi_\nu = f_\chi\left(\nu, n_e, T, n_i\right)$$

$$\eta_\nu = f_\eta\left(\nu, n_e, T, n_i\right)$$

$$\sum_{l\neq i}\left\{n_l\left[R_{li}\left(J_\nu\right) + C_{li}\right] - n_i\left(R_{il}\left(J_\nu\right) + C_{il}\right)\right\} = 0$$

$$n_i R_{il} = n_i 4\pi \int \frac{\alpha_{il}(\nu)}{h\nu} J_\nu\,\mathrm{d}\nu$$

$$n_l R_{li} = n_l \frac{g_i}{g_j} 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu}\left(\frac{2h\nu^3}{c^2} + J_\nu\right)\mathrm{d}\nu$$

Two-level atom

$$\frac{\mathrm{d}^2\left[f_\nu(z)J_\nu(z)\right]}{\mathrm{d}\tau_\nu^2} = J_\nu(z) - S_\nu(z, J_{\nu'})$$

the source function may be written as (Mihalas 1978)

$$S_\nu = (1-\varepsilon)\int \varphi_{\nu'} J_{\nu'}\,\mathrm{d}\nu' + \varepsilon B_\nu$$

$$\varepsilon = \frac{\varepsilon'}{1+\varepsilon'}$$

$$\varepsilon' = \frac{C_{21}\left(1-e^{-h\nu/kT}\right)}{A_{21}}$$

Discretization

J – continuous function of z, ν

- depth ($z \rightarrow d$)
  - equidistant in $\log m$ or $\log \tau_{\rm some}$
  - $\sim$ 4 – 5 depth points per decade
- frequency ($\nu \rightarrow n$)
  - not equidistant
  - need to resolve lines
  - continuum edges
- angle ($\mu \rightarrow m$)
  - 3 directions

all quantities expressed as J_{dn}

Discretized radiative transfer equation

for a frequency point n and angle m, depth points $d = 1, \ldots, \mathrm{ND}$

$$a_d f_{d-1} J_{d-1} + (b_d f_d + 1)\, J_d + c_d f_{d+1} J_{d+1} = S_d$$

$$(b_1 f_1 + g_1 + 1)\, J_1 + c_1 f_2 J_2 = H^- + S_1$$

$$a_{\mathrm{ND}} f_{\mathrm{ND}-1} J_{\mathrm{ND}-1} + (b_{\mathrm{ND}} f_{\mathrm{ND}} + g_{\mathrm{ND}} + 1)\, J_{\mathrm{ND}} = H^+ + S_{\mathrm{ND}}$$

$$a_d = -\left[\frac{1}{2}\left(\Delta\tau_{d-\frac{1}{2}} + \Delta\tau_{d+\frac{1}{2}}\right)\Delta\tau_{d-\frac{1}{2}}\right]^{-1}$$

$$c_d = -\left[\frac{1}{2}\left(\Delta\tau_{d-\frac{1}{2}} + \Delta\tau_{d+\frac{1}{2}}\right)\Delta\tau_{d+\frac{1}{2}}\right]^{-1}$$

$$b_d = -a_d - c_d.$$

Discretized frequency integration

$$\int J(\nu)\,\mathrm{d}\nu \rightarrow \sum_{n=1}^{\mathrm{NF}} w_n J_n$$

for lines, ensure that

$$\int \phi(\nu)\,\mathrm{d}\nu = \sum_{n=1}^{\mathrm{NF}} w_n \phi_n = 1$$

if not, it is obligatory to renormalize w_n

for continuum edges place 2 frequency points (on both sides)

Discretized angle integration

$$\int I(\mu)\,\mathrm{d}\mu \rightarrow \sum_{m=1}^{\mathrm{NF}} w_m I_m$$

3 points sufficient if Gaussian quadrature is used

also necessary to check normalization

Discretized equations of statistical equilibrium

for all $d = 1, \ldots, \mathrm{ND}$, $i = 1, \ldots, \mathrm{NL}$

$$\sum_{l \neq i} \{(n_l)_d \left[(R_{li})_d + (C_{li})_d\right] - (n_i)_d \left[(R_{il})_d + (C_{il})_d\right]\} = 0$$

$$(n_i)_d (R_{il})_d = (n_i)_d 4\pi \sum_{n=1}^{\mathrm{NF}} w_n \frac{(\alpha_{il})_n}{h\nu_n} J_{dn}$$

$$(n_l)_d (R_{li})_d = (n_l)_d \frac{g_i}{g_j} 4\pi \sum_{n=1}^{\mathrm{NF}} w_n \frac{(\alpha_{il})_n}{h\nu_n} \left(\frac{2h\nu_n^3}{c^2} + J_{dn} \right)$$

Formal solution of the RTE

solution for given χ_ν and η_ν (given S_ν) – relatively simple

$$\frac{\mathrm{d}^2\left[f_\nu(z)J_\nu(z)\right]}{\mathrm{d}\tau_\nu^2} = J_\nu(z) - S_\nu(z)$$

- 2nd order differential equation
- same numerical scheme as for the Feautrier solution

crucial for the total accuracy of the whole problem solution

Λ iteration

radiative transfer equation for J_ν

$$\frac{\mathrm{d}^2\left[f_\nu(z)J_\nu(z)\right]}{\mathrm{d}\tau_\nu^2} = J_\nu(z) - S_\nu(z)$$

formally may be written as

$$J_\nu = \Lambda_\nu S_\nu$$

iteration scheme: $J_\nu \xrightarrow{\mathrm{ESE}} S_\nu \xrightarrow{\mathrm{RTE}} J_\nu \xrightarrow{\mathrm{ESE}} S_\nu \rightarrow \cdots$

$$\sum_{l\neq i}\left\{n_l^{(n)}\left[R_{li}(J_\nu^{(n)}) + C_{li}\right] - n_i^{(n)}\left[R_{il}(J_\nu^{(n)}) + C_{il}\right]\right\} = 0$$

$$J_\nu^{(n+1)} = \Lambda_\nu S_\nu^{(n)}$$

converges extremely slowly for stellar atmospheres

Complete linearization

to stellar atmospheres introduced by Auer & Mihalas (1969) for the case of NLTE model atmospheres

solution of equations:

- radiative transfer (J_ν)
- hydrostatic equilibrium (ρ)
- radiative equilibrium (T)
- statistical equilibrium (n_i)

Complete linearization

to stellar atmospheres introduced by Auer & Mihalas (1969)
for the case of NLTE model atmospheres
for a restricted problem – NLTE line formation (Auer 1973)
solution of equations:

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solution of equations:

- radiative transfer (J_ν)

- statistical equilibrium (n_i)

vector of solution $\vec{\psi} = (J_1, \ldots, J_{\rm NF}, n_1, \ldots, n_{\rm NL})$

Complete linearization

vector of solution $\vec{\psi} = (J_1, \ldots, J_{\mathrm{NF}}, n_1, \ldots, n_{\mathrm{NL}})$, dimension $\mathrm{NF} + \mathrm{NL}$

formally $\vec{F}\left(\vec{\psi}\right) = 0$

current estimate $\vec{\psi}_0$

correct solution $\vec{\psi} = \vec{\psi}_0 + \delta\vec{\psi}$

corrections

$$\delta\vec{\psi} = -\left[\frac{\partial \vec{F}}{\partial \vec{\psi}}\left(\vec{\psi}_0\right)\right]^{-1} \cdot \vec{F}\left(\vec{\psi}_0\right)$$

matrix $\mathrm{NL} + \mathrm{NF}$ for each depth point d

Complete linearization

radiative transfer equation (from Mihalas, 1978, Stellar atmospheres)

$$\frac{f_{d-1,n}\delta J_{d-1,n}}{\Delta\tau_{d-1/2,n}\Delta\tau_{dn}} - \left[\frac{f_{dn}}{\Delta\tau_{dn}}\left(\frac{1}{\Delta\tau_{d-1/2,n}} + \frac{1}{\Delta\tau_{d+1/2,n}}\right) + \right.$$
$$\left. + \left(1 - \frac{n_{e,d}\sigma_e}{\chi_{dn}}\right)\right]\delta J_{dn} + \frac{f_{d+1,n}\delta J_{d+1,n}}{\Delta\tau_{d+1/2,n}\Delta\tau_{dn}} +$$
$$+ a_{dn}\delta\omega_{d-1,n} + b_{dn}\delta\omega_{dn} + c_{dn}\delta\omega_{d+1,n} - (\eta_{dn} + n_{e,d}\sigma_e J_d) \times$$
$$\times \frac{\delta\chi_{dn}}{\chi_{dn}^2} + \frac{\delta\eta_{dn}}{\chi_{dn}} + \frac{\sigma_e J_{dn}}{\chi_{dn}}\delta n_{e,d} =$$
$$= \beta_{dn} + (J_{dn} - n_{e,d}\sigma_e J_{dn} + \eta_{dn})/\chi_{dn}, \quad (7.39)$$

$$\alpha_{dn} = (f_{dn}J_{dn} - f_{d-1,n}J_{d-1,n})/(\Delta\tau_{d-1/2,n}\Delta\tau_{dn}), \quad (7.40)$$
$$\gamma_{dn} = (f_{dn}J_{dn} - f_{d+1,n}J_{d+1,n})/(\Delta\tau_{d+1/2,n}\Delta\tau_{dn}), \quad (7.41)$$
$$\beta_{dn} = \alpha_{dn} + \gamma_{dn}, \quad (7.42)$$
$$a_{dn} = [\alpha_{dn} + \tfrac{1}{2}\beta_{dn}(\Delta\tau_{d-1/2,n}/\Delta\tau_{dn})]/(\omega_{d-1,n} + \omega_{dn}), \quad (7.43)$$
$$c_{dn} = [\gamma_{dn} + \tfrac{1}{2}\beta_{dn}(\Delta\tau_{d+1/2,n}/\Delta\tau_{dn})]/(\omega_{dn} + \omega_{d+1,n}), \quad (7.44)$$
$$b_{dn} = a_{dn} + c_{dn}, \quad (7.45)$$
$$\omega_{dn} = \chi_{dn}/\rho_d. \quad (7.46)$$

Complete linearization

radiative transfer equation

$$\mathbb{A}_d \delta \vec{J}_{d-1} + \mathbb{B}_d \delta \vec{J}_d + \mathbb{C}_d \delta \vec{J}_{d+1} = \vec{L}_d$$

$\vec{J} = (J_1, \ldots, J_{\mathrm{NF}})$

$$\delta S_{nd} = \sum_{l=1}^{\mathrm{NL}} \left. \frac{\partial S_n}{\partial n_l} \right|_d (\delta n_l)_d$$

Complete linearization

radiative transfer equation

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$$\delta S_{nd} = \sum_{l=1}^{\mathrm{NL}} \left.\frac{\partial S_n}{\partial n_l}\right|_d (\delta n_l)_d$$

$$\mathbb{A}_d \delta J_{n,d-1} + \mathbb{B}_d \delta J_{n,d} + \mathbb{C}_d \delta J_{n,d+1} + \mathbb{D}_d \delta n_{l,d} = L_{n,d}$$

Complete linearization

radiative transfer equation

$$\mathbb{A}_d \delta J_{n,d-1} + \mathbb{B}_d \delta J_{n,d} + \mathbb{C}_d \delta J_{n,d+1} + \mathbb{D}_d \delta n_{l,d} = L_{n,d}$$

statistical equilibrium $\mathcal{A} \cdot \vec{n} = \vec{b}$, $\vec{n} = (n_1, \ldots, n_{\mathrm{NL}})$

$$\left[\frac{\partial \mathcal{A}}{\partial \vec{n}} \vec{n} - \frac{\partial b}{\partial \vec{n}} + \mathcal{A}\right]_d \delta \vec{n}_d + \left[\frac{\partial \mathcal{A}}{\partial J_{n,d}} \vec{n} - \frac{\partial b}{\partial J_{n,d}}\right] \delta J_{n,d} = \vec{b}_d - \mathcal{A}_d \cdot \vec{n}_d$$

$$\mathbb{E}_d \delta n_{l,d} + \mathbb{F}_d \delta J_{n,d} = K_{l,d}$$

Complete linearization

radiative transfer equation

$$\mathbb{A}_d \delta J_{n,d-1} + \mathbb{B}_d \delta J_{n,d} + \mathbb{C}_d \delta J_{n,d+1} + \mathbb{D}_d \delta n_{l,d} = L_{n,d}$$

statistical equilibrium

$$\mathbb{E}_d \delta n_{l,d} + \mathbb{F}_d \delta J_{n,d} = K_{l,d}$$

Complete linearization

radiative transfer equation

$$\mathbb{A}_d \delta J_{n,d-1} + \mathbb{B}_d \delta J_{n,d} + \mathbb{C}_d \delta J_{n,d+1} + \mathbb{D}_d \delta n_{l,d} = L_{n,d}$$

statistical equilibrium

$$\mathbb{E}_d \delta n_{l,d} + \mathbb{F}_d \delta J_{n,d} = K_{l,d}$$

$$\delta n_{l,d} = \sum_{n=1}^{\mathrm{NF}} \left.\frac{\partial n_l}{\partial J_n}\right|_d \delta(J_n)_d$$

$$\frac{\partial n_l}{\partial J_n} = \sum_{r=1}^{\mathrm{NL}} \mathcal{A}_{lr}^{-1} \left[\frac{\partial b_r}{\partial J_n} - \sum_{s=1}^{\mathrm{NL}} \frac{\partial \mathcal{A}_{rs}}{\partial J_n} \cdot n_s \right]$$

Complete linearization

radiative transfer $+$ statistical equilibrium

$$\mathbb{A}'_d \delta \vec{J}_{d-1} + \mathbb{B}'_d \delta \vec{J}_d + \mathbb{C}'_d \delta \vec{J}_{d+1} = \vec{L}'_d$$

$\vec{J}_d = (J_1, \ldots, J_{\mathrm{NF}})$, $d = 1, \ldots, \mathrm{ND}$
$\mathbb{A}'_1 = 0$, $\mathbb{C}'_{\mathrm{ND}} = 0$

Complete linearization

reformulation using net radiative bracket (Auer & Heasley 1976)

$$(\delta Z_t)_d = (n_i)_d(\delta R_{il})_d - (n_l)_d(\delta R_{li})_d$$

t – transition index

$$\delta n_i = \sum_t \frac{\partial n_i}{\partial Z_t}\delta Z_t$$

$$\delta \vec{Z}_t + \mathbf{R}_t \vec{n}_i + \mathbf{S}_t \delta \vec{n}_l = \vec{\mathcal{L}}_t$$

$$\left(\mathbf{1} + \mathbf{R}_t \frac{\partial n_i}{\partial Z_t} + \mathbf{S}_t \frac{\partial n_l}{\partial Z_t}\right)\delta \mathbf{Z}_t = \vec{\mathcal{L}}_t - \mathbf{R}_t \sum_{t'\neq t} \frac{\partial n_i}{\partial Z_{t'}}\delta \mathbf{Z}_{t'} - \mathbf{S}_t \sum_{t'\neq t} \frac{\partial n_l}{\partial Z_{t'}}\delta \mathbf{Z}_{t'}$$

system is solved for $(Z_t)_d$

Accelerated lambda iteration

huge matrices in complete linearization – laborious and computationally expensive
simple Λ-iteration – convergence problems
way out –> Accelerated lambda iteration (ALI)

(Cannon 1973)

formal solution

$$J_\nu = \Lambda_\nu S_\nu$$

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formal solution

$$J_\nu = \Lambda^*_\nu S_\nu + \Lambda_\nu S_\nu - \Lambda^*_\nu S_\nu$$

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formal solution

$$J_\nu = \Lambda^*_\nu S_\nu + \left(\Lambda_\nu - \Lambda^*_\nu\right) S_\nu$$

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formal solution

$$J_\nu = \Lambda^*_\nu S_\nu + \left(\Lambda_\nu - \Lambda^*_\nu\right) S_\nu$$

iteration scheme

$$J_\nu^{(n+1)} = \Lambda^*_\nu S_\nu^{(n+1)} + \left(\Lambda_\nu - \Lambda^*_\nu\right) S_\nu^{(n)}$$

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formal solution

$$J_\nu = \Lambda^*_\nu S_\nu + \left(\Lambda_\nu - \Lambda^*_\nu\right) S_\nu$$

iteration scheme

$$J_\nu^{(n+1)} = \Lambda^*_\nu S_\nu^{(n+1)} + \underbrace{\left(\Lambda_\nu - \Lambda^*_\nu\right) S_\nu^{(n)}}_{\Delta J_\nu^{(n)}}$$

correction term

Accelerated lambda iteration

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(Cannon 1973)

formal solution

$$J_\nu = \Lambda^*_\nu S_\nu + (\Lambda_\nu - \Lambda^*_\nu)\, S_\nu$$

iteration scheme

$$J_\nu^{(n+1)} = \Lambda^*_\nu S_\nu^{(n+1)} + \underbrace{(\Lambda_\nu - \Lambda^*_\nu)\, S_\nu^{(n)}}_{\Delta J_\nu^{(n)}}$$

correction term

compare: $J_\nu^{(n+1)} = \Lambda_\nu S_\nu^{(n)}$

Accelerated lambda iteration

iteration scheme

$$J_\nu^{(n+1)} = \Lambda_\nu^* S_\nu^{(n+1)} + \left(\Lambda_\nu - \Lambda_\nu^*\right) S_\nu^{(n)} = \Lambda_\nu^* S_\nu^{(n+1)} + \Delta J_\nu^{(n)}$$

$\Delta J_\nu^{(n)}$ – correction term

$$S_\nu \xrightarrow{\text{RTE}} J_\nu, \Delta J_\nu \xrightarrow{\text{ESE+ALI}} S_\nu \rightarrow \cdots$$

solution of the radiative transfer equation is transferred to the solution of the statistical equilibrium equations

Accelerated lambda iteration

iteration scheme

$$J_\nu^{(n+1)} = \Lambda_\nu^* S_\nu^{(n+1)} + \left(\Lambda_\nu - \Lambda_\nu^*\right) S_\nu^{(n)} = \Lambda_\nu^* S_\nu^{(n+1)} + \Delta J_\nu^{(n)}$$

$J_\nu^{(n+1)} \longrightarrow$ radiative rates

$$n_i R_{il} = n_i 4\pi \int \frac{\alpha_{il}(\nu)}{h\nu} \left[\Lambda_\nu^* S_\nu^{(n+1)} + \Delta J_\nu^{(n)}\right] \mathrm{d}\nu$$

$$n_l R_{li} = n_l \frac{g_i}{g_j} 4\pi \int \frac{\alpha_{ij}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + \left[\Lambda_\nu^* S_\nu^{(n+1)} + \Delta J_\nu^{(n)}\right]\right) \mathrm{d}\nu$$

Accelerated lambda iteration

equations of statistical equilibrium

$$\sum_{l\neq i}\{n_l\left[R_{li}\left(n_i,n_l\right)+C_{li}\right]-n_i\left(R_{il}\left(n_i,n_l\right)+C_{il}\right)\}=0$$

for $i=1,\ldots \mathrm{NL}$

- ALI eliminated radiation field from the explicit solution
- equations of statistical equilibrium are nonlinear in n_i, n_l
- savings important, for a typical problem $\mathrm{NF}\sim 10000$

Accelerated lambda iteration

equations of statistical equilibrium

$$\sum_{l \neq i} \{ n_l \left[R_{li} \left(n_i, n_l \right) + C_{li} \right] - n_i \left(R_{il} \left(n_i, n_l \right) + C_{il} \right) \} = 0$$

solution by

- Newton-Raphson method (linearization)
- preconditioning

Summary

two basic methods of the solution of the RTE+ESE system

1. complete linearization (Kiel)
2. accelerated lambda iteration (DETAIL)